2.2 Conditional Statements

Difficulty Level: **Basic** | Created by: CK-12

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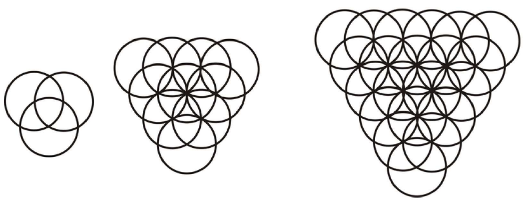
Learning Objectives

- Identify the hypothesis and conclusion of an if-then or conditional statement.
- Write the converse, inverse, and contrapositive of an if-then statement.
- Recognize a biconditional statement.

Review Queue

Find the next figure or term in the pattern.

1. 5, 8, 12, 17, 23,...
2. $\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{9}, \frac{6}{10}, \ldots$



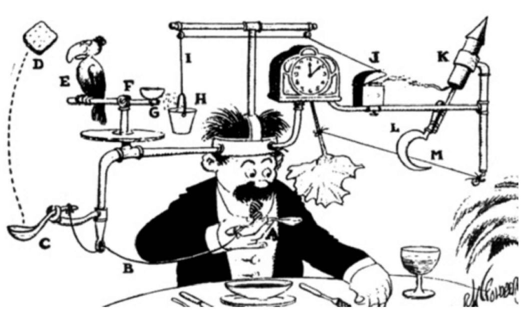
[Figure 1]

3. Find a counterexample for the following conjectures.

   a. If it is April, then it is Spring Break.

   b. If it is June, then I am graduating.

Know What? Rube Goldberg was a cartoonist in the 1940s who drew crazy inventions to do very simple things. The invention to the right has a series of smaller tasks that leads to the machine wiping the man's face with a napkin.



[Figure 2]

Write a series of if-then statements to that would caption this cartoon, from A to M.

If-Then Statements

Conditional Statement (also called an **If-Then Statement**): A statement with a hypothesis followed by a conclusion.

Another way to define a conditional statement is to say, "If this happens, then that will happen."

Hypothesis: The first, or "if," part of a conditional statement. An educated guess.

Conclusion: The second, or "then," part of a conditional statement. The conclusion is the result of a hypothesis.

Keep in mind that conditional statements might not always be written in the "if-then" form. Here are a few examples.

Statement 1: If you work overtime, then you'll be paid time-and-a-half.

Statement 2: I'll wash the car if the weather is nice.

Statement 3: If 2 divides evenly into x, then x is an even number.

Statement 4: I'll be a millionaire when I win monopoly.

Statement 5: All equiangular triangles are equilateral.

Statements 1 and 3 are written in the "if-then" form. The hypothesis of Statement 1 is "you work overtime." The conclusion is "you'll be paid time-and-a-half."

So, if Sarah works overtime, then what will happen? From Statement 1, we can conclude that she will be paid time-and-a-half.

If 2 goes evenly into 16, what can you conclude? From Statement 3, we know that 16 must be an even number.

Statement 2 has the hypothesis after the conclusion. Even though the word "then" is not there, the statement can be rewritten as: If the weather is nice, then I'll wash the car. If the word "if" is in the middle of a conditional statement, the hypothesis is always after it.

Statement 4 uses the word "when" instead of "if." It should be treated like Statement 2, so it can be written as: If I win monopoly, then I will be a millionaire.

Statement 5 "if" and "then" are not there, but can be rewritten as: If a triangle is equiangular, then it is equilateral.

Converse, Inverse, and Contrapositive of a Conditional Statement

Look at **Statement 2** again: *If the weather is nice, then I'll wash the car.*

This can be rewritten using letters to represent the hypothesis and conclusion.

$$\text{If } p, \text{then } q. \qquad p = \text{the weather is nice}$$
$$q = \text{I'll wash the car}$$
$$\text{Or}, p \rightarrow q$$

In addition to these positives, we can also write the negations, or "not"s of p and q. The symbolic version of not p, is $\sim p$.

$$\sim p = \text{the weather is not nice}$$
$$\sim q = \text{I won't wash the car}$$

Using these negations and switching the order of p and q, we can create three more conditional statements.

Converse	$q \rightarrow p$	$\underbrace{\text{If I wash the car}}_{q}, \underbrace{\text{then the weather is nice}}_{p}.$
Inverse	$\sim p \rightarrow \sim q$	$\underbrace{\text{If the weather is not nice}}_{\sim p}, \underbrace{\text{then I won't wash the}}_{\sim q}$
Contrapositive	$\sim q \rightarrow \sim p$	$\underbrace{\text{If I don't wash the car}}_{\sim q}, \underbrace{\text{then the weather is not n}}_{\sim p}$

If we accept "If the weather is nice, then I'll wash the car" as true, then the converse and inverse are not necessarily true. However, if we take original statement to be true, then the contrapositive is also true. We say that the contrapositive is ***logically equivalent*** to the original if-then statement.

Example 1: Use the statement: If $n > 2$, then $n^2 > 4$.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

Converse :	If $n^2 > 4$, then $n > 2$.	*False.* n could be -3, making $n^2 =$
Inverse :	If $n < 2$, then $n^2 < 4$.	*False.* Again, if $n = -3$, then $n^2 =$
Contrapositive :	If $n^2 < 4$, then $n < 2$.	*True,* the only square number less t 4 is 1, which has square roots of 1 or less than 2.

Example 2: Use the statement: If I am at Disneyland, then I am in California.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

Converse :	If I am in California, then I am at Disneyland. *False.* I could be in San Francisco.
Inverse :	If I am not at Disneyland, then I am not in California. *False.* Again, I could be in San Francisco.
Contrapositive :	If I am not in California, then I am not at Disneyland. *True.* If I am not in the state, I couldn't be at Disneyland.

Notice for the inverse and converse ***we can use the same counterexample.*** This is because the inverse and converse are also ***logically equivalent.***

Example 3: Use the statement: Any two points are collinear.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: First, change the statement into an "if-then" statement: If two points are on the same line, then they are collinear.

$\underline{\text{Converse}}$: If two points are collinear, then they are on the same line. $True$

$\underline{\text{Inverse}}$: If two points are not on the same line, then they are not collinea

$\underline{\text{Contrapositive}}$: If two points are not collinear, then they do not lie on the same

Biconditional Statements

Example 3 is an example of a biconditional statement.

Biconditional Statement: When the original statement and converse are both true.

So, $p \rightarrow q$ is true and $q \rightarrow p$ is true. It is written $p \leftrightarrow q$, with a double arrow to indicate that it does not matter if p or q is first. It is said, "p if and only if q"

Example 4: Rewrite Example 3 as a biconditional statement.

Solution: *If two points are on the same line, then they are collinear* can be rewritten as: *Two points are on the same line <u>if and only if</u> they are collinear.*

Replace the "if-then" with "if and only if" in the middle of the statement. "If and only if" can be abbreviated "iff."

Example 5: The following is a true statement:

$m\angle ABC > 90^\circ$ if and only if $\angle ABC$ is an obtuse angle.

Determine the two true statements within this biconditional.

Solution:

<u>Statement 1</u>: If $m\angle ABC > 90^\circ$, then $\angle ABC$ is an obtuse angle

<u>Statement 2:</u> If $\angle ABC$ is an obtuse angle, then $m\angle ABC > 90^\circ$.

You should recognize this as the definition of an obtuse angle. All geometric definitions are biconditional statements.

Example 6: $p : x < 10 \qquad q : 2x < 50$

a) Is $p \rightarrow q$ true? If not, find a counterexample.

b) Is $q \rightarrow p$ true? If not, find a counterexample.

c) Is $\sim p \rightarrow \sim q$ true? If not, find a counterexample.

d) Is $\sim q \rightarrow \sim p$ true? If not, find a counterexample.

Solution:

$$\begin{array}{lll} p \rightarrow q: & \text{If } x < 10, \text{ then } 2x < 50. & True. \\ q \rightarrow p: & \text{If } 2x < 50, \text{ then } x < 10. & False, \ x = 15 \text{ would be a counterexample} \\ \sim p \rightarrow \sim q: & \text{If } x > 10, \text{ then } 2x > 50. & False, \ x = 15 \text{ would also work here.} \\ \sim q \rightarrow \sim p: & \text{If } 2x > 50, \text{ then } x > 10. & True. \end{array}$$

Know What? Revisited The conditional statements are as follows:

$A \rightarrow B$: If the man raises his spoon, then it pulls a string.

$B \rightarrow C$: If the string is pulled, then it tugs back a spoon.

$C \rightarrow D$: If the spoon is tugged back, then it throws a cracker into the air.

$D \rightarrow E$: If the cracker is tossed into the air, the bird will eat it.

$E \rightarrow F$: If the bird eats the cracker, then it turns the pedestal.

$F \rightarrow G$: If the bird turns the pedestal, then the water tips over.

$G \rightarrow H$: If the water tips over, it goes into the bucket.

$H \rightarrow I$: If the water goes into the bucket, then it pulls down the string.

$I \rightarrow J$: If the bucket pulls down the string, then the string opens the box.

$J \rightarrow K$: If the box is opened, then a fire lights the rocket.

$K \rightarrow L$: If the rocket is lit, then the hook pulls a string.

$L \rightarrow M$: If the hook pulls the string, then the man's faces is wiped with the napkin.

This is a very complicated contraption used to wipe a man's face. Purdue University liked these cartoons so much, that they started the Rube Goldberg Contest in 1949. This past year, the task was to pump hand sanitizer into someone's hand in no less than 20 steps. http://www.purdue.edu/newsroom/rubegoldberg/index.html

Review Questions

For questions 1-6, determine the hypothesis and the conclusion.

1. If 5 divides evenly into x, then x ends in 0 or 5.
2. If a triangle has three congruent sides, it is an equilateral triangle.
3. Three points are coplanar if they all lie in the same plane.
4. If $x = 3$, then $x^2 = 9$.
5. If you take yoga, then you are relaxed.
6. All baseball players wear hats.
7. Write the converse, inverse, and contrapositive of #1. Determine if they are true or false. If they are false, find a counterexample.
8. Write the converse, inverse, and contrapositive of #5. Determine if they are true or false. If they are false, find a counterexample.
9. Write the converse, inverse, and contrapositive of #6. Determine if they are true or false. If they are false, find a counterexample.
10. Find the converse of #2. If it is true, write the biconditional of the statement.
11. Find the converse of #3. If it is true, write the biconditional of the statement.
12. Find the converse of #4. If it is true, write the biconditional of the statement.

For questions 13-16, use the statement: If $AB = 5$ and $BC = 5$, then B is the midpoint of $\overline{AC}$.

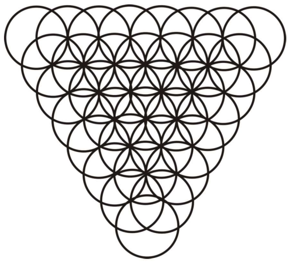
13. If this is the converse, what is the original statement? Is it true?
14. If this is the original statement, what is the inverse? Is it true?
15. Find a counterexample of the statement.
16. Find the contrapositive of the original statement from #13.
17. What is the inverse of the inverse of $p \rightarrow q$? HINT: Two wrongs make a right in math!
18. What is the one-word name for the converse of the inverse of an if-then statement?
19. What is the one-word name for the inverse of the converse of an if-then statement?
20. What is the contrapositive of the contrapositive of an if-then statement?

For questions 21-24, determine the two true conditional statements from the given biconditional statements.

21. A U.S. citizen can vote if and only if he or she is 18 or more years old.
22. A whole number is prime if and only if it has exactly two distinct factors.
23. Points are collinear if and only if there is a line that contains the points.
24. $2x = 18$ if and only if $x = 9$.
25. $p : x = 4 \quad q : x^2 = 16$
    a. Is $p \rightarrow q$ true? If not, find a counterexample.
    b. Is $q \rightarrow p$ true? If not, find a counterexample.
    c. Is $\sim p \rightarrow \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \rightarrow \sim p$ true? If not, find a counterexample.
26. $p : x = -2 \quad q : -x + 3 = 5$
    a. Is $p \rightarrow q$ true? If not, find a counterexample.
    b. Is $q \rightarrow p$ true? If not, find a counterexample.
    c. Is $\sim p \rightarrow \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \rightarrow \sim p$ true? If not, find a counterexample.
27. $p :$ the measure of $\angle ABC = 90^\circ$ $q : \angle ABC$ is a right angle
    a. Is $p \rightarrow q$ true? If not, find a counterexample.
    b. Is $q \rightarrow p$ true? If not, find a counterexample.
    c. Is $\sim p \rightarrow \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \rightarrow \sim p$ true? If not, find a counterexample.
28. $p :$ the measure of $\angle ABC = 45^\circ$ $q : \angle ABC$ is an acute angle
    a. Is $p \rightarrow q$ true? If not, find a counterexample.
    b. Is $q \rightarrow p$ true? If not, find a counterexample.
    c. Is $\sim p \rightarrow \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \rightarrow \sim p$ true? If not, find a counterexample.
29. Write a conditional statement. Write the converse, inverse and contrapositive of your statement. Are they true or false? If they are false, write a counterexample.
30. Write a true biconditional statement. Separate it into the two true conditional statements.

Review Queue Answers

1. 30
2. $\frac{7}{11}$



[Figure 3]

3. Answers:

a. It could be another day that isn't during Spring Break. Spring Break doesn't last the entire month.

b. You could be a freshman, sophomore or junior. There are several counterexamples.