5.4 Medians and Altitudes in Triangles

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Learning Objectives

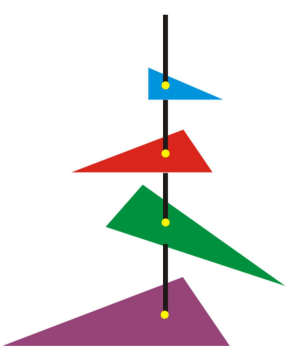
- Define median and find their point of concurrency in a triangle.
- Apply medians to the coordinate plane.
- Construct the altitude of a triangle and find their point of concurrency in a triangle.

Review Queue

1. Find the midpoint between (9, -1) and (1, 15).
2. Find the equation of the line between the two points from #1.
3. Find the equation of the line that is perpendicular to the line from #2 through (-6, 2).

Know What? Triangles are frequently used in art. Your art teacher assigns an art project involving triangles. You decide to make a series of hanging triangles of all different sizes from one long piece of wire. Where should you hang the triangles from so that they balance horizontally?

You decide to plot one triangle on the coordinate plane to find the location of this point. The coordinates of the vertices are (0, 0), (6, 12) and (18, 0). What is the coordinate of this point?

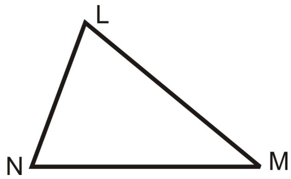


[Figure 1]

Medians

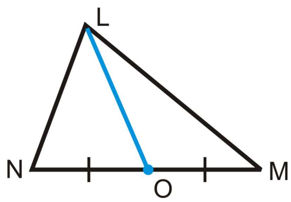
Median: The line segment that joins a vertex and the midpoint of the opposite side (of a triangle).

Example 1: Draw the median LO for $\triangle LMN$ below.



[Figure 2]

Solution: From the definition, we need to locate the midpoint of NM . We were told that the median is LO , which means that it will connect the vertex L and the midpoint of NM , to be labeled O . Measure NM and make a point halfway between N and M . Then, connect O to L .

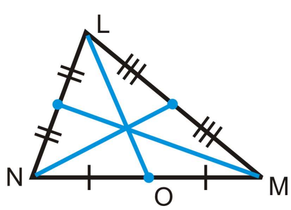


[Figure 3]

Notice that a median is very different from a perpendicular bisector or an angle bisector. A perpendicular bisector also goes through the midpoint, but it does not necessarily go through the vertex of the opposite side. And, unlike an angle bisector, a median does not necessarily bisect the angle.

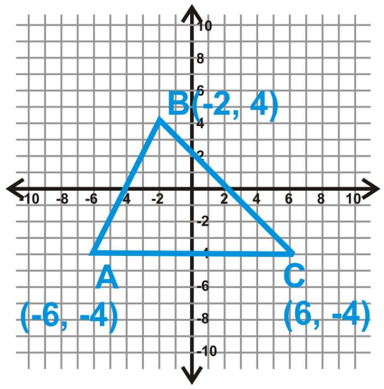
Example 2: Find the other two medians of $\triangle LMN$.

Solution: Repeat the process from Example 1 for sides LN and LM . Be sure to always include the appropriate tick marks to indicate midpoints.



[Figure 4]

Example 3: Find the equation of the median from B to the midpoint of AC for the triangle in the $x-y$ plane below.



[Figure 5]

Solution: To find the equation of the median, first we need to find the midpoint of AC , using the Midpoint Formula.

$$\left(\frac{-6+6}{2},\frac{-4+(-4)}{2}\right)=\left(\frac{0}{2},\frac{-8}{2}\right)=(0,-4)$$

Now, we have two points that make a line, B and the midpoint. Find the slope and $y-$ intercept.

$$\begin{aligned}m&=\frac{-4-4}{0-(-2)}=\frac{-8}{2}=-4\\ y&=-4x+b\\ -4&=-4(0)+b\\ -4&=b\end{aligned}$$

The equation of the median is $y=-4x-4$

Point of Concurrency for Medians

From Example 2, we saw that the three medians of a triangle intersect at one point, just like the perpendicular bisectors and angle bisectors. This point is called the centroid.

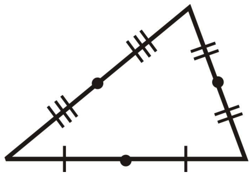
Centroid: The point of concurrency for the medians of a triangle.

Unlike the circumcenter and incenter, the centroid does not have anything to do with circles. It has a different property.

Investigation 5-3: Properties of the Centroid

Tools Needed: pencil, paper, ruler, compass

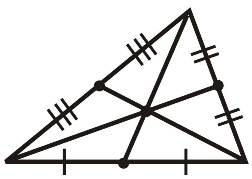
1. Construct a scalene triangle with sides of length 6 cm, 10 cm, and 12 cm (Investigation 4-2). Use the ruler to measure each side and mark the midpoint.



[Figure 6]

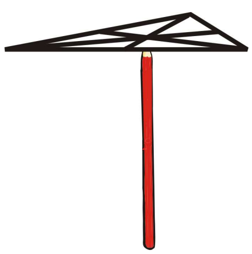
2. Draw in the medians and mark the centroid.

Measure the length of each median. Then, measure the length from each vertex to the centroid and from the centroid to the midpoint. Do you notice anything?



[Figure 7]

3. Cut out the triangle. Place the centroid on either the tip of the pencil or the pointer of the compass. What happens?



[Figure 8]

From this investigation, we have discovered the properties of the centroid. They are summarized below.

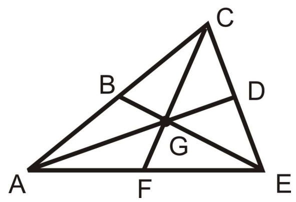
Concurrency of Medians Theorem: The medians of a triangle intersect in a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side. The centroid is also the "balancing point" of a triangle.

If G is the centroid, then we can conclude:

$$AG = \frac{2}{3}AD, CG = \frac{2}{3}CF, EG = \frac{2}{3}BE$$
$$DG = \frac{1}{3}AD, FG = \frac{1}{3}CF, BG = \frac{1}{3}BE$$

And, combining these equations, we can also conclude:

$$DG = \frac{1}{2}AG, FG = \frac{1}{2}CG, BG = \frac{1}{2}EG$$



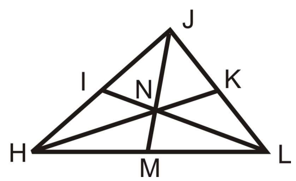
[Figure 9]

In addition to these ratios, G is also the balance point of $\triangle ACE$. This means that the triangle will balance when placed on a pencil (#3 in Investigation 5-3) at this point.

Example 4: I, K, and M are midpoints of the sides of $\triangle HJL$.

a) If $JM = 18$, find JN and NM.

b) If $HN = 14$, find NK and HK.



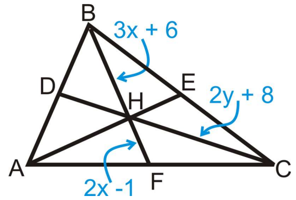
[Figure 10]

Solution:

a) JN is two-thirds of JM. So, $JN = \frac{2}{3} \cdot 18 = 12$. NM is either half of 12, a third of 18 or $18 - 12$. $NM = 6$.

b) HN is two-thirds of HK. So, $14 = \frac{2}{3} \cdot HK$ and $HK = 14 \cdot \frac{3}{2} = 21$. NK is a third of 21, half of 14, or $21 - 14$. $NK = 7$.

Example 5: ***Algebra Connection*** H is the centroid of $\triangle ABC$ and $DC = 5y - 16$. Find x and y.



[Figure 11]

Solution: HF is half of BH . Use this information to solve for x . For y , HC is two-thirds of DC . Set up an equation for both.

$$\frac{1}{2}BH = HF \text{ or } BH = 2HF \qquad HC = \frac{2}{3}DC \text{ or } \frac{3}{2}HC = DC$$

$$3x+6 = 2(2x-1) \qquad \frac{3}{2}(2y+8) = 5y-16$$

$$3x+6 = 4x-2 \qquad 3y+12 = 5y-16$$

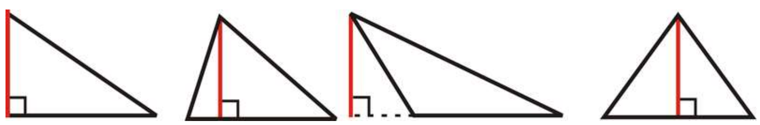
$$8 = x \qquad 14 = y$$

Altitudes

The last line segment within a triangle is an altitude. It is also called the height of a triangle.

Altitude: A line segment from a vertex and perpendicular to the opposite side.

Here are a few examples.



[Figure 12]

As you can see, an altitude can be a side of a triangle or outside of the triangle. When a triangle is a right triangle, the altitude, or height, is the leg. If the triangle is obtuse, then the altitude will be outside of the triangle. ***To construct an altitude, use Investigation 3-2*** (constructing a perpendicular line through a point not on the given line). Think of the vertex as the point and the given line as the opposite side.

Investigation 5-4: Constructing an Altitude for an Obtuse Triangle

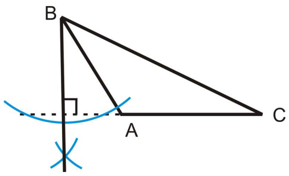
Tools Needed: pencil, paper, compass, ruler

1. Draw an obtuse triangle. Label it $\triangle ABC$, like the picture to the right. Extend side AC , beyond point A .

[Figure 13]

2. Using Investigation 3-2, construct a perpendicular line to AC , through B .

The altitude does not have to extend past side AC , as it does in the picture. Technically the height is only the vertical distance from the highest vertex to the opposite side.

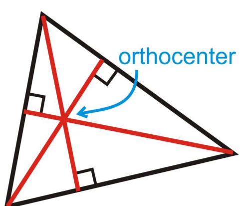
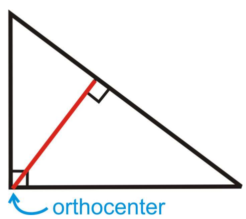
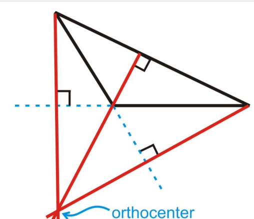


[Figure 14]

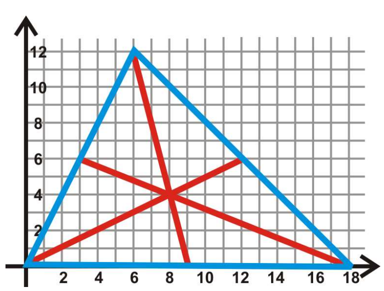
As was true with perpendicular bisectors, angle bisectors, and medians,the altitudes of a triangle are also concurrent. Unlike the other three, the point does not have any special properties.

Orthocenter: The point of concurrency for the altitudes of triangle.

Here is what the orthocenter looks like for the three triangles. It has three different locations, much like the perpendicular bisectors.

Acute Triangle	Right Triangle	Obtuse Triangle
 [Figure 15]	 [Figure 16]	 [Figure 17]
The orthocenter is inside the triangle.	The legs of the triangle are two of the altitudes. The orthocenter is the vertex of the right angle.	The orthocenter is outside the triangle.

Know What? Revisited The point that you should put the wire through is the centroid. That way, each triangle will balance on the wire.

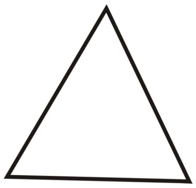


[Figure 18]

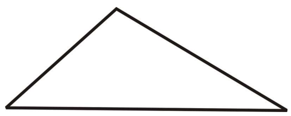
The triangle that we wanted to plot on the $x-y$ plane is to the right. Drawing all the medians, it looks like the centroid is (8, 4). To verify this, you could find the equation of two medians and set them equal to each other and solve for x. Two equations are $y=\frac{1}{2}x$ and $y=-4x+36$. Setting them equal to each other, we find that $x=8$ and then $y=4$.

Review Questions

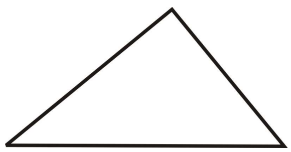
Construction Construct the centroid for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-3.

1. 

[Figure 19]

2. 

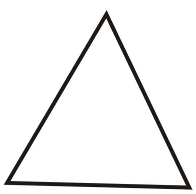
[Figure 20]

3. 

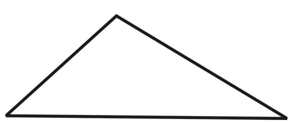
[Figure 21]

4. Is the centroid always going to be inside of the triangle? Why?

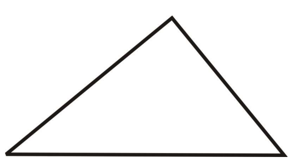
Construction Construct the orthocenter for the following triangles by tracing each triangle onto a piece of paper and using Investigations 3-2 and 5-4.

5. 

[Figure 22]

6. 

[Figure 23]

7. 

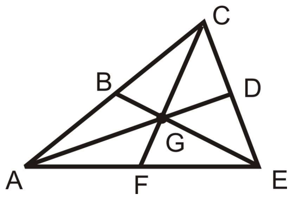
[Figure 24]

8. What do you think will happen if the triangle is equilateral? What can we say about the incenter, circumcenter, centroid, and orthocenter? Why do you think this is?
9. How many lines do you actually have to "construct" to find any point of concurrency?

For questions 10-13, find the equation of each median, from vertex A to the opposite side, BC.

10. $A(9,5), B(2,5), C(4,1)$
11. $A(-2,3), B(-3,-7), C(5,-5)$
12. $A(-1,5), B(0,-1), C(6,3)$
13. $A(6,-3), B(-5,-4), C(-1,-8)$

For questions 14-18, B, D, and F are the midpoints of each side and G is the centroid. Find the following lengths.

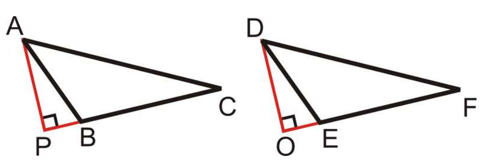


[Figure 25]

14. If $BG = 5$, find GE and BE
15. If $CG = 16$, find GF and CF
16. If $AD = 30$, find AG and GD
17. If $GF = x$, find GC and CF
18. If $AG = 9x$ and $GD = 5x - 1$, find x and AD.

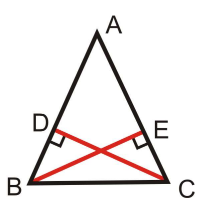
Write a two-column proof.

19. Given: $\triangle ABC \cong \triangle DEF$ AP and DO are altitudes Prove: $AP \cong DO$



[Figure 26]

20. Given: Isosceles $\triangle ABC$ with legs AB and AC $BD \perp DC$ and $CE \perp BE$ Prove: $BD \cong CE$



[Figure 27]

Use $\triangle ABC$ with $A(-2,9), B(6,1)$ and $C(-4,-7)$ for questions 21-26.

21. Find the midpoint of AB and label it M.
22. Write the equation of $\overleftrightarrow{CM}$.
23. Find the midpoint of BC and label it N.
24. Write the equation of $\overleftrightarrow{AN}$.
25. Find the intersection of $\overleftrightarrow{CM}$ and $\overleftrightarrow{AN}$.
26. What is this point called?

Another way to find the centroid of a triangle in the coordinate plane is to find the midpoint of one side and then find the point two thirds of the way from the third vertex to this point. To find the point two thirds of the way from point $A(x_1,y_1)$ to $B(x_2,y_2)$ use the formula: $\left(\frac{x_1+2x_2}{3},\frac{y_1+2y_2}{3}\right)$. Use this method to find the centroid in the following problems.

27. (-1, 3), (5, -2) and (-1, -4)
28. (1, -2), (-5, 4) and (7, 7)
29. Use the coordinates $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) and the method used in the last two problems to find a formula for the centroid of a triangle in the coordinate plane.
30. Use your formula from problem 29 to find the centroid of the triangle with vertices (2, -7), (-5, 1) and (6, -9).

Review Queue Answers

1. $midpoint = \left(\frac{9+1}{2},\frac{-1+15}{2}\right) = (5,7)$
2. $m = \frac{15+1}{1-9} = \frac{16}{-8} = -2$ $\quad 15 = -2(1)+b \quad$ $y = -2x+17$
   $17 = b$
3. $$\begin{aligned} y &= \frac{1}{2}x+b \\ 2 &= \frac{1}{2}(-6)+b \\ 2 &= -3+b \\ 5 &= b \\ y &= \frac{1}{2}x+5 \end{aligned}$$