6.3 Proving Quadrilaterals are Parallelograms

FlexBooks® 2.0 > American HS Geometry > Proving Quadrilaterals are Parallelograms

Last Modified: Dec 25, 2014

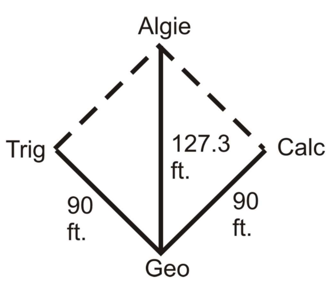
Learning Objectives

- Prove a quadrilateral is a parallelogram using the converses of the theorems from the previous section.
- Prove a quadrilateral is a parallelogram in the coordinate plane.

Review Queue

1. Write the converses of: the Opposite Sides Theorem, Opposite Angles Theorem, Consecutive Angles Theorem and the Parallelogram Diagonals Theorem.
2. Are any of these converses true? If not, find a counterexample.
3. Plot the points $A(2,2),\ B(4,-2),\ C(-2,-4)$, and $D(-6,-2)$.
   a. Find the slopes of $\overline{AB},\ \overline{BC},\ \overline{CD}$, and $\overline{AD}$. Is $ABCD$ a parallelogram?
   b. Find the point of intersection of the diagonals. Does this go along with what you found in part a?

Know What? Four friends, Geo, Trig, Algie, and Calc are marking out a baseball diamond. Geo is standing at home plate. Trig is 90 feet away at 3^{rd} base, Algie is 127.3 feet away at 2^{nd} base, and Calc is 90 feet away at 1^{st} base. The angle at home plate is 90° , from 1^{st} to 3^{rd} is 90° . Find the length of the other diagonal and determine if the baseball diamond is a parallelogram. If it is, what kind of parallelogram is it?



[Figure 1]

Determining if a Quadrilateral is a Parallelogram

In the last section, we introduced the Opposite Sides Theorem, Opposite Angles Theorem, Consecutive Angles Theorem and the Parallelogram Diagonals Theorem. #1 in the Review Queue above, had you write the converses of each of these:

Opposite Sides Theorem Converse: If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

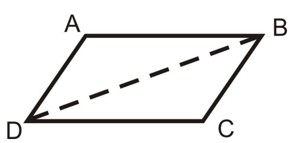
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse: If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

Are these converses true? The answer is yes. Each of these converses can be a way to show that a quadrilateral is a parallelogram. However, the Consecutive Angles Converse can be a bit tricky, considering you would have to show that each angle is supplementary to its neighbor $(\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle A$ and $\angle D)$. We will not use this converse.

Proof of the Opposite Sides Theorem Converse



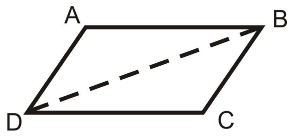
[Figure 2]

Given: $AB \cong DC$, $AD \cong BC$

Prove: $ABCD$ is a parallelogram

Statement	***Reason***
1. $AB \cong DC$, $AD \cong BC$	Given
2. $DB \cong DB$	Reflexive PoC
3. $\triangle ABD \cong \triangle CDB$	SSS
4. $\angle ABD \cong \angle BDC$, $\angle ADB \cong \angle DBC$	CPCTC
5. $AB \parallel DC$, $AD \parallel BC$	Alternate Interior Angles Converse
6. $ABCD$ is a parallelogram	Definition of a parallelogram

Example 1: Write a two-column proof.



[Figure 3]

Given: $AB || DC$ and $AB \cong DC$

Prove: $ABCD$ is a parallelogram

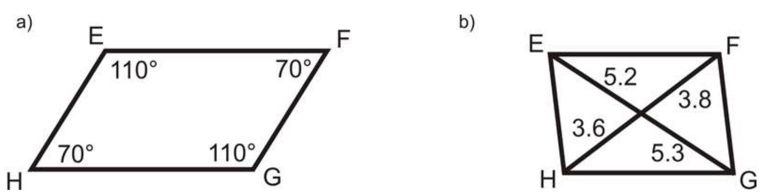
Solution:

Statement	***Reason***
1. $AB || DC$ and $AB \cong DC$	Given
2. $\angle ABD \cong \angle BDC$	Alternate Interior Angles
3. $DB \cong DB$	Reflexive PoC
4. $\triangle ABD \cong \triangle CDB$	SAS
5. $AD \cong BC$	CPCTC
6. $ABCD$ is a parallelogram	Opposite Sides Converse

Example 1 proves an additional way to show that a quadrilateral is a parallelogram.

Theorem 5-10: If a quadrilateral has one set of parallel lines that are also congruent, then it is a parallelogram.

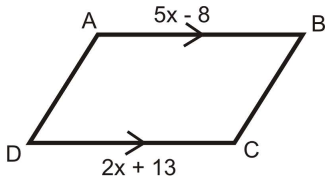
Example 2: Is quadrilateral $EFGH$ a parallelogram? How do you know?



[Figure 4]

Solution: For part a, the opposite angles are equal, so by the Opposite Angles Theorem Converse, $EFGH$ is a parallelogram. In part b, the diagonals do not bisect each other, so $EFGH$ is not a parallelogram.

Example 3: ***Algebra Connection*** What value of x would make $ABCD$ a parallelogram?



[Figure 5]

Solution: $AB || DC$ from the markings. By Theorem 5-10, $ABCD$ would be a parallelogram if $AB = DC$ as well.

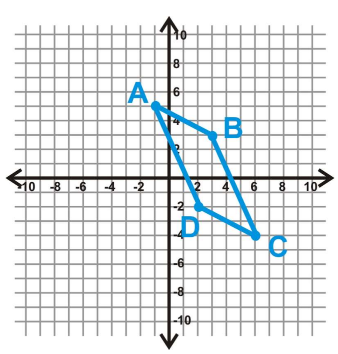
$$5x - 8 = 2x + 13$$
$$3x = 21$$
$$x = 7$$

In order for $ABCD$ to be a parallelogram, x must equal 7.

Showing a Quadrilateral is a Parallelogram in the Coordinate Plane

To show that a quadrilateral is a parallelogram in the $x-y$ plane, you will need to use a combination of the slope formulas, the distance formula and the midpoint formula. For example, to use the Definition of a Parallelogram, you would need to ***find the slope of all four sides*** to see if the opposite sides are parallel. To use the Opposite Sides Converse, you would have to find the length (***using the distance formula***) of each side to see if the opposite sides are congruent. To use the Parallelogram Diagonals Converse, you would need to use the ***midpoint formula*** for each diagonal to see if the midpoint is the same for both. Finally, you can use Theorem 5-10 in the coordinate plane. To use this theorem, you would need to show that one pair of opposite sides has the same slope (***slope formula***) and the same length (***distance formula***).

Example 4: Is the quadrilateral $ABCD$ a parallelogram?



[Figure 6]

Solution: We have determined there are four different ways to show a quadrilateral is a parallelogram in the $x-y$ plane. Let's use Theorem 5-10. First, find the length of AB and

CD .

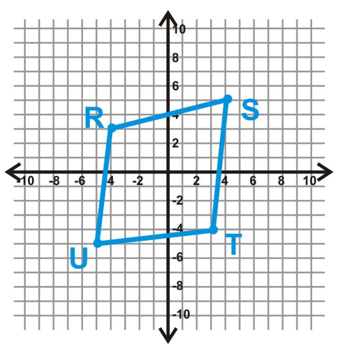
$$\begin{aligned} AB &= \sqrt{(-1-3)^2+(5-3)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned} \qquad \begin{aligned} CD &= \sqrt{(2-6)^2+(-2+4)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned}$$

$AB = CD$, so if the two lines have the same slope, $ABCD$ is a parallelogram.

Slope $AB = \frac{5-3}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$ Slope $CD = \frac{-2+4}{2-6} = \frac{2}{-4} = -\frac{1}{2}$

By Theorem 5-10, $ABCD$ is a parallelogram.

Example 5: Is the quadrilateral $RSTU$ a parallelogram?



[Figure 7]

Solution: Let's use the Parallelogram Diagonals Converse to determine if $RSTU$ is a parallelogram. Find the midpoint of each diagonal.

Midpoint of $RT = \left(\frac{-4+3}{2}, \frac{3-4}{2}\right) = (-0.5, -0.5)$

Midpoint of $SU = \left(\frac{4-5}{2}, \frac{5-5}{2}\right) = (-0.5, 0)$

Because the midpoint is not the same, $RSTU$ is not a parallelogram.

Know What? Revisited First, we can use the Pythagorean Theorem to find the length of the second diagonal.

$$\begin{aligned} 90^2 + 90^2 &= d^2 \\ 8100 + 8100 &= d^2 \\ 16200 &= d^2 \\ d &= 127.3 \end{aligned}$$

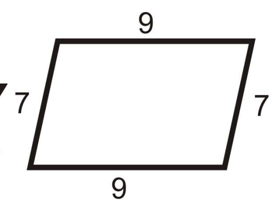
This means that the diagonals are equal. If the diagonals are equal, the other two sides of the diamond are also 90 feet. Therefore, the baseball diamond is a parallelogram, and more specifically, it is a square.

[Figure 8]

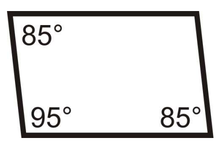
Review Questions

For questions 1-12, determine if the quadrilaterals are parallelograms. If they are, write a reason.

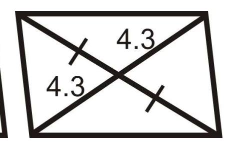
[Figure 9]



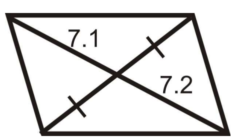
[Figure 10]



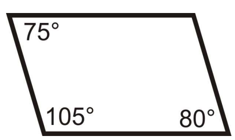
[Figure 11]



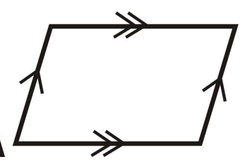
[Figure 12]

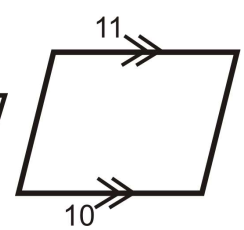


[Figure 13]

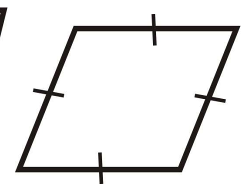


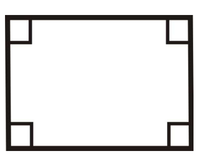
[Figure 14]

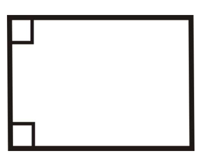

[Figure 15]

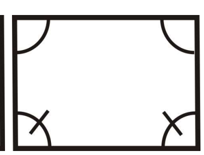


[Figure 16]

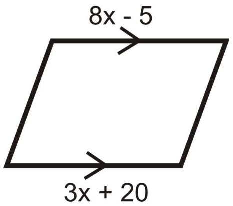

[Figure 17]


[Figure 18]

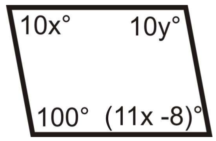

[Figure 19]


[Figure 20]

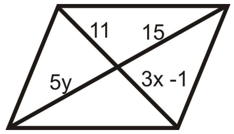
For questions 13-15, determine the value of x and y that would make the quadrilateral a parallelogram.



[Figure 21]



[Figure 22]



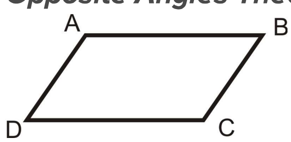
[Figure 23]

For questions 16-18, determine if $ABCD$ is a parallelogram.

16. $A(8,-1),\ B(6,5),\ C(-7,2),\ D(-5,-4)$
17. $A(-5,8),\ B(-2,9),\ C(3,4),\ D(0,3)$
18. $A(-2,6),\ B(4,-4),\ C(13,-7),\ D(4,-10)$

Write a two-column proof.

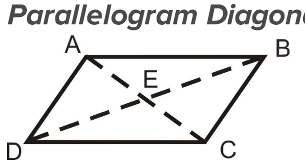
Opposite Angles Theorem Converse



[Figure 24]

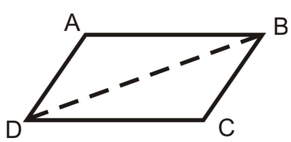
Given: $\angle A \cong \angle C,\ \angle D \cong \angle B$ Prove: $ABCD$ is a parallelogram

Parallelogram Diagonals Theorem Converse



[Figure 25]

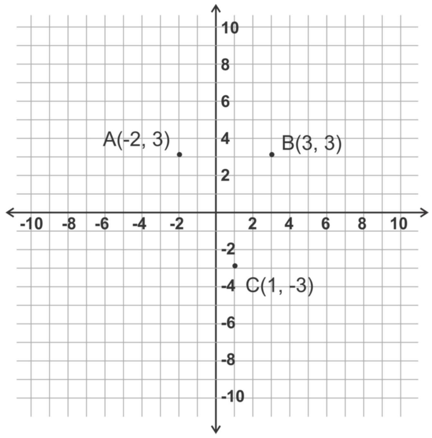
Given: $AE \cong EC,\ DE \cong EB$ Prove: $ABCD$ is a parallelogram



[Figure 26]

Given: $\angle ADB \cong CBD,\ AD \cong BC$ Prove: $ABCD$ is a parallelogram

Suppose that $A(-2,3),\ B(3,3)$ and $C(1,-3)$ are three of four vertices of a parallelogram.



[Figure 27]

22. Depending on where you choose to put point D, the name of the parallelogram you draw will change. Sketch a picture to show all possible parallelograms. How many can you draw?
23. If you know the parallelogram is named $ABDC$, what is the slope of side parallel to AC ?
24. Again, assuming the parallelogram is named $ABDC$, what is the length of BD?
25. Find the points of intersection of the diagonals of the three parallelograms formed. Label them X in parallelogram $ABCD$, Y in parallelogram $ADBC$ and Z in parallelogram $ABDC$.
26. Connect the points X, Y and Z to form a triangle. What do you notice about this triangle?

The points $Q(-1,1), U(7,1), A(1,7)$ and $D(-1,5)$ are the vertices of quadrilateral $QUAD$. Plot the points on graph paper to complete problems 27-30.

27. Find the midpoints of sides QU, UA, AD and DQ. Label them W, X, Y and Z respectively.
28. Connect the midpoints to form quadrilateral $WXYZ$. What does this quadrilateral appear to be?
29. Use slopes to verify your answer to problem 29.
30. Use midpoints to verify your answer to problem 29.
31. This phenomenon occurs in all quadrilaterals. Describe how you might prove this fact. (Hint: each side of quadrilateral $WXYZ$ is a midsegment in a triangle formed by two sides of the parallelogram and a diagonal.)

Review Queue Answers

1. **Opposite Sides Theorem Converse:** If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

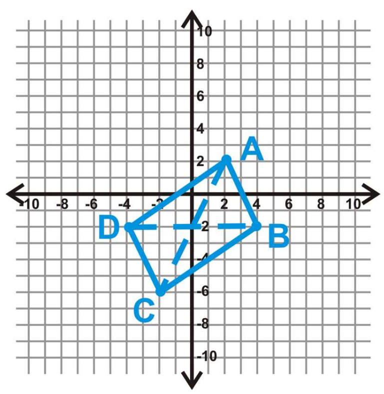
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse: If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

2. All the converses are true.

3.



[Figure 28]

a) $$\text{Slope } AB = \text{Slope } CD = -\frac{1}{2}$$
$$\text{Slope } AD = \text{Slope } BC = \frac{2}{3}$$

$ABCD$ is a parallelogram because the opposite sides are parallel.

b) $$\text{Midpoint of } BD = (0, -2)$$
$$\text{Midpoint of } AC = (0, -2)$$

Yes, the midpoint of the diagonals are the same, so they bisect each other. This corresponds with what we found in part a.