11.6 Surface Area and Volume of Spheres

FlexBooks® 2.0 > American HS Geometry > Surface Area and Volume of Spheres

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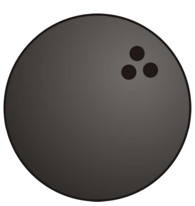
Learning Objectives

- Find the surface area of a sphere.
- Find the volume of a sphere.

Review Queue

1. List three spheres you would see in real life.
2. Find the area of a circle with a 6 cm radius.
3. Find the volume of a cylinder with the circle from #2 as the base and a height of 5 cm.
4. Find the volume of a cone with the circle from #2 as the base and a height of 5 cm.

Know What? A regulation bowling ball is a sphere that weighs between 12 and 16 pounds. The maximum circumference of a bowling ball is 27 inches. Using this number, find the radius of a bowling ball, its surface area and volume. You may assume the bowling ball does not have any finger holes. Round your answers to the nearest hundredth.



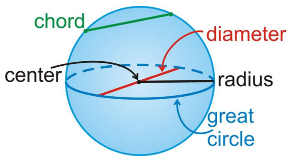
[Figure 1]

Defining a Sphere

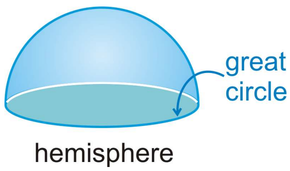
A sphere is the last of the three-dimensional shapes that we will find the surface area and volume of. Think of a sphere as a three-dimensional circle. You have seen spheres in real-life countless times; tennis balls, basketballs, volleyballs, golf balls, and baseballs. Now we will analyze the parts of a sphere.

Sphere: The set of all points, in three-dimensional space, which are equidistant from a point.

A sphere has a ***center***, radius and diameter, just like a circle. The ***radius*** has an endpoint on the sphere and the other is on the center. The ***diameter*** must contain the center. If it does not, it is a ***chord***. The ***great circle*** is a plane that contains the diameter. It would be the largest circle cross section in a sphere. There are infinitely many great circles. ***The circumference of a sphere is the circumference of a great circle***. Every great circle divides a sphere into two congruent hemispheres, or two half spheres. Also like a circle, spheres can have tangent lines and secants. These are defined just like they are in a circle.



[Figure 2]



[Figure 3]

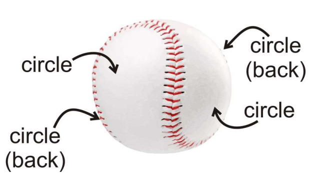
Example 1: The circumference of a sphere is $26\pi\ feet$. What is the radius of the sphere?

Solution: The circumference is referring to the circumference of a great circle. Use $C = 2\pi r$.

$$\begin{aligned} 2\pi r &= 26\pi \\ r &= 13\ ft. \end{aligned}$$

Surface Area of a Sphere

One way to find the formula for the surface area of a sphere is to look at a baseball. We can best *approximate* the cover of the baseball by 4 circles. The area of a circle is πr^2 , so the surface area of a sphere is $4\pi r^2$. While the covers of a baseball are not four perfect circles, they are stretched and skewed.



[Figure 4]

Another way to show the surface area of a sphere is to watch the link by Russell Knightley, http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Surface-Area-Derivation.html. It is a great visual interpretation of the formula.

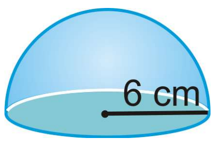
Surface Area of a Sphere: If r is the radius, then the surface area of a sphere is $SA = 4\pi r^2$.

Example 2: Find the surface area of a sphere with a radius of 14 feet.

Solution: Use the formula, $r = 14\ ft$.

$$\begin{aligned}SA &= 4\pi(14)^2 \\ &= 784\pi\ ft^2\end{aligned}$$

Example 3: Find the surface area of the figure below.



[Figure 5]

Solution: This is a hemisphere. Be careful when finding the surface area of a hemisphere because you need to include the area of the base. If the question asked for the *lateral surface area*, then your answer would *not* include the bottom.

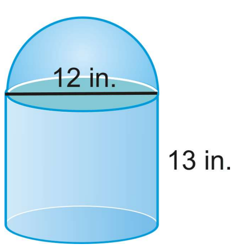
$$\begin{aligned}SA &= \pi r^2 + \frac{1}{2}4\pi r^2 \\ &= \pi(6^2) + 2\pi(6^2) \\ &= 36\pi + 72\pi = 108\pi\ cm^2\end{aligned}$$

Example 4: The surface area of a sphere is $100\pi\ in^2$. What is the radius?

Solution: Plug in what you know to the formula and solve for r .

$$\begin{aligned}100\pi &= 4\pi r^2 \\ 25 &= r^2 \\ 5 &= r\end{aligned}$$

Example 5: Find the surface area of the following solid.



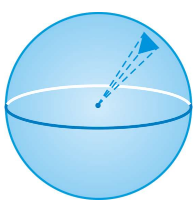
[Figure 6]

Solution: This solid is a cylinder with a hemisphere on top. Because it is one fluid solid, we would not include the bottom of the hemisphere or the top of the cylinder because they are no longer on the surface of the solid. Below, " LA " stands for *lateral area*.

$$\begin{aligned} SA &= LA_{cylinder} + LA_{hemisphere} + A_{base\ circle} \\ &= \pi r h + \frac{1}{2} 4\pi r^2 + \pi r^2 \\ &= \pi(6)(13) + 2\pi 6^2 + \pi 6^2 \\ &= 78\pi + 72\pi + 36\pi \\ &= 186\pi\ in^2 \end{aligned}$$

Volume of a Sphere

A sphere can be thought of as a regular polyhedron with an infinite number of congruent regular polygon faces. As n , the number of faces increases to an infinite number, the figure approaches becoming a sphere. So a sphere can be thought of as a polyhedron with an infinite number faces. Each of those faces is the base of a pyramid whose vertex is located at the center of the sphere. Each of the pyramids that make up the sphere would be congruent to the pyramid shown. The volume of this pyramid is given by $V = \frac{1}{3} Bh$.



[Figure 7]

To find the volume of the sphere, you need to add up the volumes of an infinite number of infinitely small pyramids.

$$\begin{aligned} V(all\ pyramids) &= V_1 + V_2 + V_3 + \ldots + V_n \\ &= \frac{1}{3}(B_1 h + B_2 h + B_3 h + \ldots + B_n h) \\ &= \frac{1}{3} h(B_1 + B_2 + B_3 + \ldots + B_n) \end{aligned}$$

The sum of all of the bases of the pyramids is the surface area of the sphere. Since you know that the surface area of the sphere is $4\pi r^2$, you can substitute this quantity into the equation above.

$$= \frac{1}{3}h\left(4\pi r^2\right)$$

In the picture above, we can see that the height of each pyramid is the radius, so $h = r$.

$$= \frac{4}{3}r(\pi r^2)$$
$$= \frac{4}{3}\pi r^3$$

To see an animation of the volume of a sphere, see http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Volume-Derivation.html by Russell Knightley. It is a slightly different interpretation than our derivation.

Volume of a Sphere: If a sphere has a radius r , then the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Example 6: Find the volume of a sphere with a radius of 9 m.

Solution: Use the formula above.

$$V = \frac{4}{3}\pi 6^3$$
$$= \frac{4}{3}\pi(216)$$
$$= 288\pi$$

Example 7: A sphere has a volume of $14137.167\ ft^3$, what is the radius?

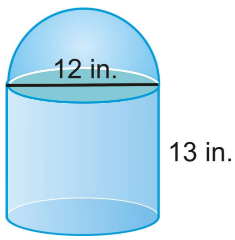
Solution: Because we have a decimal, our radius might be an approximation. Plug in what you know to the formula and solve for r .

$$14137.167 = \frac{4}{3}\pi r^3$$
$$\frac{3}{4\pi}\cdot 14137.167 = r^3$$
$$3375 = r^3$$

At this point, you will need to take the ***cubed root*** of 3375. Depending on your calculator, you can use the $\sqrt[3]{x}$ function or $\wedge\left(\frac{1}{3}\right)$. The cubed root is the inverse of cubing a number, just like the square root is the inverse, or how you undo, the square of a number.

$$\sqrt[3]{3375} = 15 = r \qquad \text{The radius is } 15\ ft.$$

Example 8: Find the volume of the following solid.



[Figure 8]

Solution: To find the volume of this solid, we need the volume of a cylinder and the volume of the hemisphere.

$$V_{cylinder} = \pi 6^2(13) = 78\pi$$
$$V_{hemisphere} = \frac{1}{2}\left(\frac{4}{3}\pi 6^3\right) = 36\pi$$
$$V_{total} = 78\pi + 36\pi = 114\pi\ in^3$$

Know What? Revisited If the maximum circumference of a bowling ball is 27 inches, then the maximum radius would be $27 = 2\pi r$, or $r = 4.30$ inches. Therefore, the surface area would be $4\pi 4.3^2 \approx 232.35\ in^2$, and the volume would be $\frac{4}{3}\pi\, 4.3^3 \approx 333.04\ in^3$. The weight of the bowling ball refers to its density, how heavy something is. The volume of the ball tells us how much it can hold.

Review Questions

1. Are there any cross-sections of a sphere that are not a circle? Explain your answer.

Find the surface area and volume of a sphere with: (Leave your answer in terms of π)

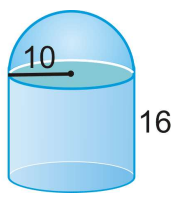
2. a radius of 8 in.
3. a diameter of 18 cm.
4. a radius of 20 ft.
5. a diameter of 4 m.
6. a radius of 15 ft.
7. a diameter of 32 in.
8. a circumference of $26\pi\ cm$.

9. a circumference of $50\pi\, yds$.
10. The surface area of a sphere is $121\pi\, in^2$. What is the radius?
11. The volume of a sphere is $47916\pi\, m^3$. What is the radius?
12. The surface area of a sphere is $4\pi\, ft^2$. What is the volume?
13. The volume of a sphere is $36\pi\, mi^3$. What is the surface area?
14. Find the radius of the sphere that has a volume of $335\, cm^3$. Round your answer to the nearest hundredth.
15. Find the radius of the sphere that has a surface area $225\pi\, ft^2$.

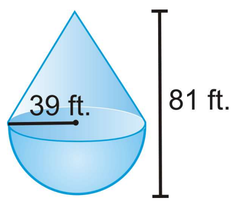
Find the surface area of the following shapes. Leave your answers in terms of π .

You may assume the bottom is *open*.

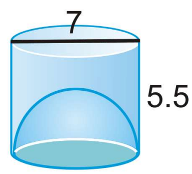
[Figure 9]



[Figure 10]



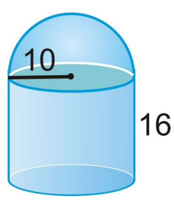
[Figure 11]



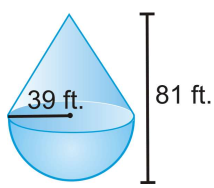
[Figure 12]

Find the volume of the following shapes. Round your answers to the nearest hundredth.

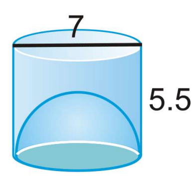
[Figure 13]



[Figure 14]



[Figure 15]



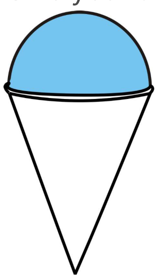
[Figure 16]

20. A sphere has a radius of 5 cm. A right cylinder has the same radius and volume. Find the height and total surface area of the cylinder.

Tennis balls with a 3 inch diameter are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space *not* occupied by the tennis balls? Round your answer to the nearest hundredth.

[Figure 17]

One hot day at a fair you buy yourself a snow cone. The height of the cone shaped container is 5 cm and its radius is 2 cm. The shaved ice is perfectly rounded on top forming a hemisphere. What is the volume of the ice in your frozen treat? If the ice melts at a rate of 2 cm^3 per minute, how long do you have to eat your treat before it all melts? Give your answer to the nearest minute.



[Figure 18]

Multi-Step Problems

27. Answer the following:
    a. What is the surface area of a cylinder?
    b. Adjust your answer from part a for the case where $r = h$.
    c. What is the surface area of a sphere?
    d. What is the relationship between your answers to parts b and c? Can you explain this?
28. At the age of 81, Mr. Luke Roberts began collecting string. He had a ball of string 3 feet in diameter.
    a. Find the volume of Mr. Roberts' ball of string in cubic inches.
    b. Assuming that each cubic inch weighs 0.03 pounds, find the weight of his ball of string.
    c. To the nearest inch, how big (diameter) would a 1 ton ball of string be? $(1\ ton = 2000\ lbs)$

For problems 29-31, use the fact that the earth's radius is approximately 4,000 miles.

29. Find the length of the equator.
30. Find the surface area of earth, rounding your answer to the nearest million square miles.
31. Find the volume of the earth, rounding your answer to the nearest billion cubic miles.

Review Queue Answers

1. *Answers will vary.* Possibilities are any type of ball, certain lights, or the 76/Unical orb.
2. 36π
3. 180π
4. 60π