4.4 Triangle Congruence Using ASA, AAS, and HL

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Learning Objectives

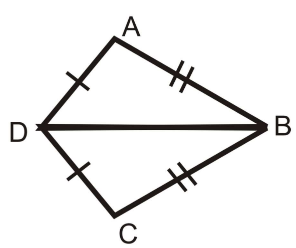
- Use the ASA Congruence Postulate, AAS Congruence Theorem, and the HL Congruence Theorem.
- Complete two-column proofs using SSS, SAS, ASA, AAS, and HL.

Review Queue

1. Write a two-column proof.

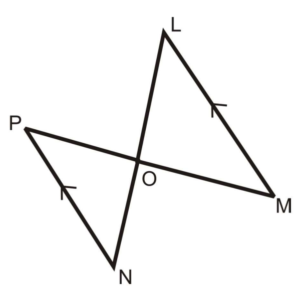
Given: $\overline{AD} \cong \overline{DC}, \overline{AB} \cong \overline{CB}$

Prove: $\triangle DAB \cong \triangle DCB$



[Figure 1]

2. Is $\triangle PON \cong \triangle MOL$? Why or why not?



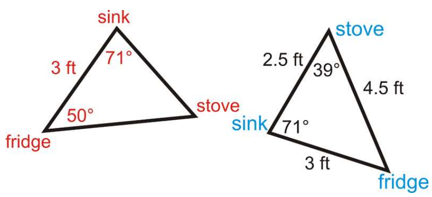
[Figure 2]

3. If $\triangle DEF \cong \triangle PQR$, can it be assumed that:

a) $\angle F \cong \angle R$? Why or why not?

b) $EF \cong PR$? Why or why not?

Know What? Your parents changed their minds at the last second about their kitchen layout. Now, they have decided they to have the distance between the sink and the fridge be 3 ft, the angle at the sink 71° and the angle at the fridge is 50° . You used your protractor to measure the angle at the stove and sink at your neighbor's house. Are the kitchen triangles congruent now?



[Figure 3]

ASA Congruence

Like SAS, ASA refers to Angle-Side-Angle. The placement of the word Side is important because it indicates that the side that you are given is between the two angles.

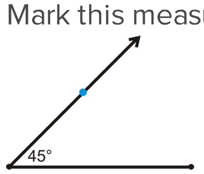
Consider the question: If I have two angles that are 45° and 60° and the side between them is 5 in, can I construct only one triangle? We will investigate it here.

Investigation 4-4: Constructing a Triangle Given Two Angles and Included Side Tools Needed: protractor, pencil, ruler, and paper

Draw the side (5 in) horizontally, halfway down the page. *The drawings in this investigation are to scale.*

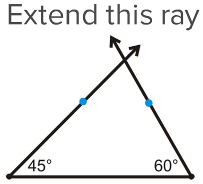
[Figure 4]

At the left endpoint of your line segment, use the protractor to measure the 45° angle. Mark this measurement and draw a ray from the left endpoint through the 45° mark.



[Figure 5]

At the right endpoint of your line segment, use the protractor to measure the 60° angle. Mark this measurement and draw a ray from the left endpoint through the 60° mark. Extend this ray so that it crosses through the ray from Step 2.



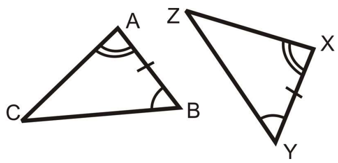
[Figure 6]

1. Erase the extra parts of the rays from Steps 2 and 3 to leave only the triangle.

Can you draw another triangle, with these measurements that looks different? The answer is NO. ***Only one triangle can be created from any given two angle measures and the INCLUDED side.***

Angle-Side-Angle (ASA) Congruence Postulate: If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent.

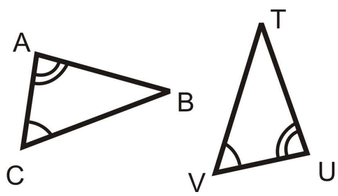
The markings in the picture are enough to say $\triangle ABC \cong \triangle XYZ$.



[Figure 7]

Now, in addition to SSS and SAS, you can use ASA to prove that two triangles are congruent.

Example 1: What information would you need to prove that these two triangles are congruent using the ASA Postulate?



[Figure 8]

a) $AB \cong UT$

b) $AC \cong UV$

c) $BC \cong TV$

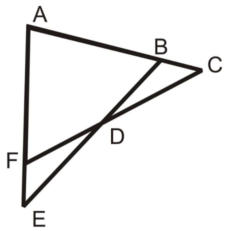
d) $\angle B \cong \angle T$

Solution: For ASA, we need the side between the two given angles, which is AC and UV . The answer is b.

Example 2: Write a 2-column proof.

Given: $\angle C \cong \angle E, AC \cong AE$

Prove: $\triangle ACF \cong \triangle AEB$

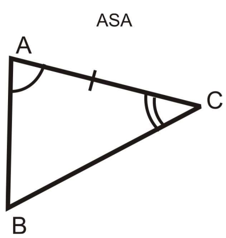


[Figure 9]

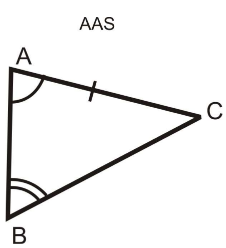
Statement	*Reason*
1. $\angle C \cong \angle E, AC \cong AE$	Given
2. $\angle A \cong \angle A$	Reflexive PoC
3. $\triangle ACF \cong \triangle AEB$	ASA

AAS Congruence

A variation on ASA is AAS, which is Angle-Angle-Side. Recall that for ASA you need two angles and the side between them. But, if you know two pairs of angles are congruent, then the third pair will also be congruent by the 3^{rd} Angle Theorem. Therefore, you can prove a triangle is congruent whenever you have any two angles and a side.



[Figure 10]



[Figure 11]

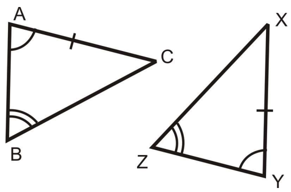
Be careful to note the placement of the side for ASA and AAS. As shown in the pictures above, the side is ***between*** the two angles for ASA and it is not for AAS.

Angle-Angle-Side (AAS or SAA) Congruence Theorem: If two angles and a non-included side in one triangle are congruent to two corresponding angles and a non-included side in another triangle, then the triangles are congruent.

Proof of AAS Theorem:

Given: $\angle A \cong \angle Y, \angle B \cong \angle Z, AC \cong XY$

Prove: $\triangle ABC \cong \triangle YZX$



[Figure 12]

Statement	***Reason***
1. $\angle A \cong \angle Y, \angle B \cong \angle Z, AC \cong XY$	Given
2. $\angle C \cong \angle X$	3^{rd} Angle Theorem
3. $\triangle ABC \cong \triangle YZX$	ASA

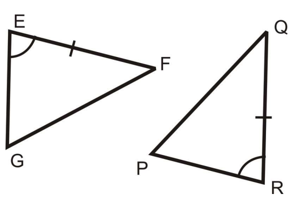
By proving $\triangle ABC \cong \triangle YZX$ with ASA, we have also shown that the AAS Theorem is valid. You can now use this theorem to show that two triangles are congruent.

Example 3: What information do you need to prove that these two triangles are congruent using:

a) ASA?

b) AAS?

c) SAS?



[Figure 13]

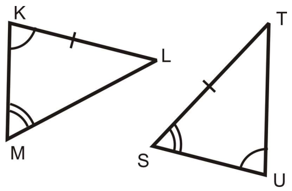
Solution:

a) For ASA, we need the angles on the other side of EF and QR. Therefore, we would need $\angle F \cong \angle Q$.

b) For AAS, we would need the angle on the other side of $\angle E$ and $\angle R$. $\angle G \cong \angle P$.

c) For SAS, we would need the side on the other ***side*** of $\angle E$ and $\angle R$. So, we would need $EG \cong RP$.

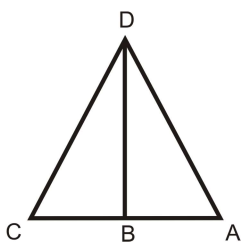
Example 4: Can you prove that the following triangles are congruent? Why or why not?



[Figure 14]

Solution: Even though $KL \cong ST$, they are not corresponding. Look at the angles around $KL, \angle K$ and $\angle L$. $\angle K$ has **one** arc and $\angle L$ is unmarked. The angles around ST are $\angle S$ and $\angle T$. $\angle S$ has **two** arcs and $\angle T$ is unmarked. In order to use AAS, $\angle S$ needs to be congruent to $\angle K$. They are not congruent because the arcs marks are different. Therefore, we cannot conclude that these two triangles are congruent.

Example 5: Write a 2-column proof.



[Figure 15]

Given: BD is an angle bisector of $\angle CDA, \angle C \cong \angle A$

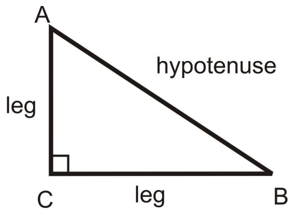
Prove: $\triangle CBD \cong \angle ABD$

Solution:

Statement	***Reason***
1. BD is an angle bisector of $\angle CDA, \angle C \cong \angle A$	Given
2. $\angle CDB \cong \angle ADB$	Definition of an Angle Bisector
3. $DB \cong DB$	Reflexive PoC
3. $\triangle CBD \cong \triangle ABD$	AAS

Hypotenuse-Leg Congruence Theorem

So far, the congruence postulates we have learned will work on any triangle. The last congruence theorem can only be used on right triangles. Recall that a right triangle has exactly one right angle. The two sides adjacent to the right angle are called legs and the side opposite the right angle is called the hypotenuse.



[Figure 16]

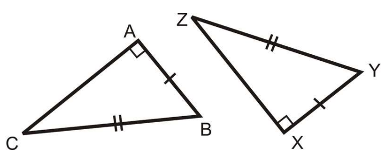
You may or may not know the Pythagorean Theorem (which will be covered in more depth later in this text). It says, for any ***right*** triangle, this equation is true:

$(leg)^2 + (leg)^2 = (hypotenuse)^2$. What this means is that if you are given two sides of a right triangle, you can always find the third.

Therefore, if you know that two sides of a ***right*** triangle are congruent to two sides of another ***right*** triangle, you can conclude that third sides are also congruent.

HL Congruence Theorem: If the hypotenuse and leg in one right triangle are congruent to the hypotenuse and leg in another right triangle, then the two triangles are congruent.

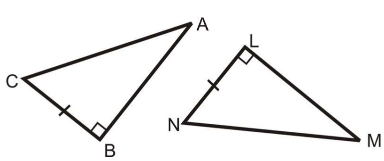
The markings in the picture are enough to say $\triangle ABC \cong \triangle XYZ$.



[Figure 17]

Notice that this theorem is only used with a hypotenuse and a leg. If you know that the two legs of a right triangle are congruent to two legs of another triangle, the two triangles would be congruent by SAS, because the right angle would be between them. We will not prove this theorem here because we have not proven the Pythagorean Theorem yet.

Example 6: What information would you need to prove that these two triangles are congruent using the: a) HL Theorem? b) SAS Theorem?



[Figure 18]

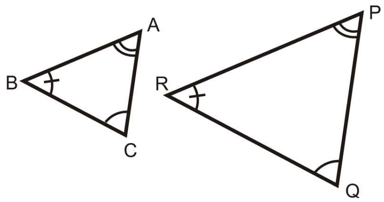
Solution:

a) For HL, you need the hypotenuses to be congruent. So, $AC \cong MN$.

b) To use SAS, we would need the other legs to be congruent. So, $AB \cong ML$.

AAA and SSA Relationships

There are two other side-angle relationships that we have not discussed: AAA and SSA.

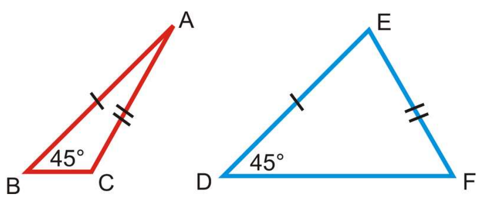


[Figure 19]

AAA implied that all the angles are congruent, however, that does not mean the triangles are congruent.

As you can see, $\triangle ABC$ and $\triangle PRQ$ are not congruent, even though all the angles are. These triangles are similar, a topic that will be discussed later in this text.

SSA relationships do not prove congruence either. In review problems 29 and 30 of the last section you illustrated an example of how SSA could produce two different triangles. $\triangle ABC$ and $\triangle DEF$ below are another example of SSA.



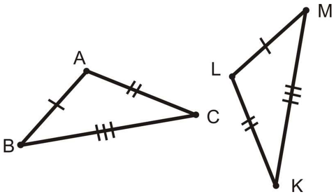
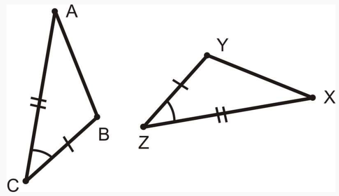
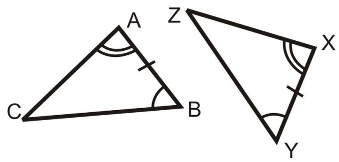
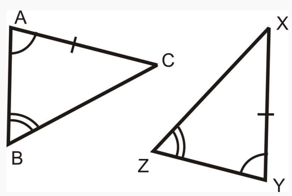
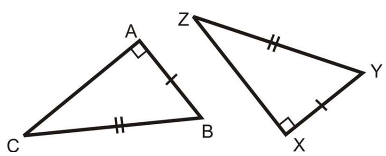
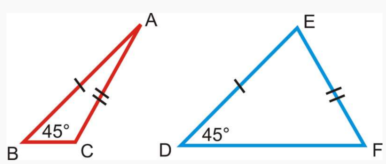
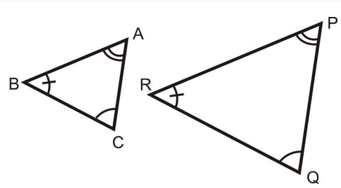
[Figure 20]

$\angle B$ and $\angle D$ are ***not*** the included angles between the congruent sides, so we cannot prove that these two triangles are congruent. Notice, that two different triangles can be drawn even though $AB \cong DE$, $AC \cong EF$, and $m\angle B = m\angle D$.

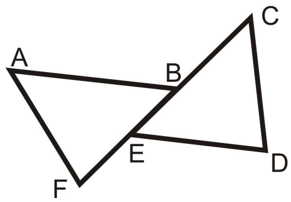
You might have also noticed that SSA could also be written ASS. This is true, however, in this text we will write SSA.

Triangle Congruence Recap

To recap, here is a table of all of the possible side-angle relationships and if you can use them to determine congruence or not.

Side-Angle Relationship	*Picture*	*Determine Congruence?*
SSS	 [Figure 21]	**Yes** $\triangle ABC \cong \triangle LKM$
SAS	 [Figure 22]	**Yes** $\triangle ABC \cong \triangle XYZ$
ASA	 [Figure 23]	**Yes** $\triangle ABC \cong \triangle XYZ$
AAS (or SAA)	 [Figure 24]	**Yes** $\triangle ABC \cong \triangle YZX$
HL	 [Figure 25]	**Yes, Right Triangles Only** $\triangle ABC \cong \triangle XYZ$
SSA (or ASS)	 [Figure 26]	**NO**
AAA	 [Figure 27]	**NO**

Example 7: Write a 2-column proof.



[Figure 28]

Given: $AB \parallel ED, \angle C \cong \angle F, AB \cong ED$

Prove: $AF \cong CD$

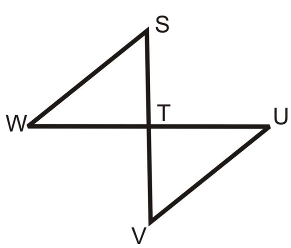
Solution:

Statement	***Reason***
1. $AB \parallel ED, \angle C \cong \angle F, AB \cong ED$	Given
2. $\angle ABE \cong \angle DEB$	Alternate Interior Angles Theorem
3. $\triangle ABF \cong \triangle DEC$	ASA
4. $AF \cong CD$	CPCTC

Example 8: Write a 2-column proof.

Given: T is the midpoint of WU and SV

Prove: $WS \parallel VU$



[Figure 29]

Solution:

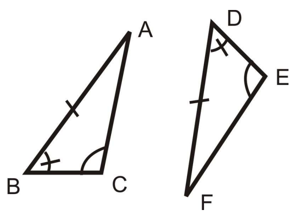
Statement	***Reason***
1. T is the midpoint of WU and SV	Given
2. $WT \cong TU$, $ST \cong TV$	Definition of a midpoint
3. $\angle STW \cong \angle UTV$	Vertical Angle Theorem
4. $\triangle STW \cong \triangle VTU$	SAS
5. $\angle S \cong \angle V$	CPCTC
6. $WS \parallel VU$	Converse of the Alternate Interior Angles Theorem

Prove Move: At the beginning of this chapter we introduced CPCTC. Now, it can be used in a proof once two triangles are proved congruent. It is used to prove the parts of congruent triangles are congruent in order to prove that sides are parallel (like in Example 8), midpoints, or angle bisectors. You will do proofs like these in the review questions.

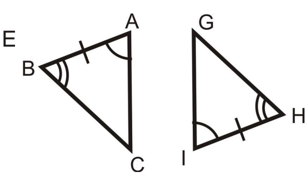
Know What? Revisited Even though we do not know all of the angle measures in the two triangles, we can find the missing angles by using the Third Angle Theorem. In your parents' kitchen, the missing angle is 39° . The missing angle in your neighbor's kitchen is 50° . From this, we can conclude that the two kitchens are now congruent, either by ASA or AAS.

Review Questions

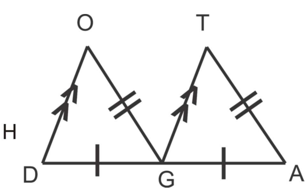
For questions 1-14, determine if the triangles are congruent. If they are, write the congruence statement and which congruence postulate or theorem you used.



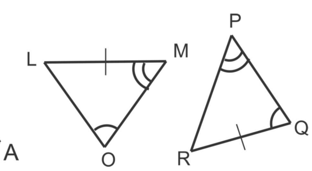
[Figure 30]



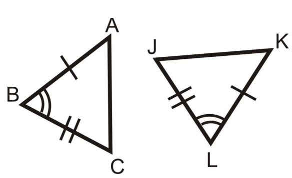
[Figure 31]



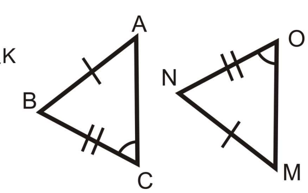
[Figure 32]



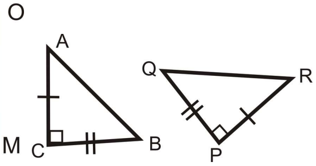
[Figure 33]



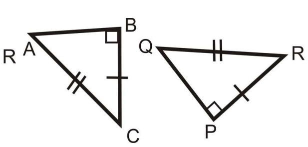
[Figure 34]



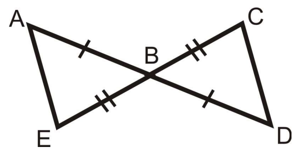
[Figure 35]



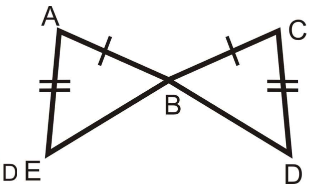
[Figure 36]



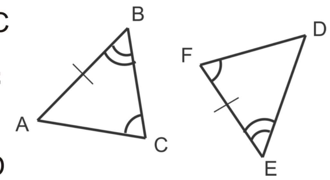
[Figure 37]



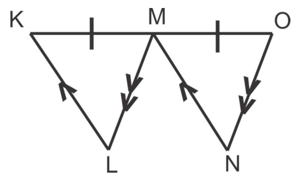
[Figure 38]



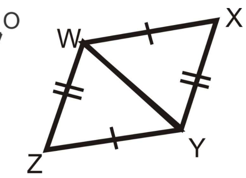
[Figure 39]



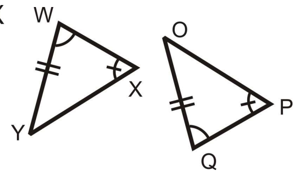
[Figure 40]



[Figure 41]

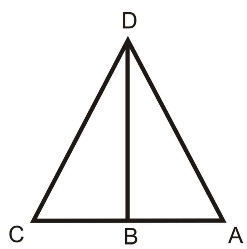


[Figure 42]



[Figure 43]

For questions 15-19, use the picture and the given information below.



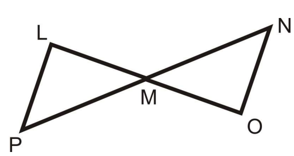
[Figure 44]

<u>Given</u>: $DB \perp AC, DB$ is the angle bisector of $\angle CDA$

15. From $DB \perp AC$, which angles are congruent and why?
16. Because DB is the angle bisector of $\angle CDA$, what two angles are congruent?
17. From looking at the picture, what additional piece of information are you given? Is this enough to prove the two triangles are congruent?
18. Write a 2-column proof to prove $\triangle CDB \cong \triangle ADB$.
19. What would be your reason for $\angle C \cong \angle A$?

For questions 20-24, use the picture to the right and the given information.

<u>Given</u>: $LP \parallel NO, LP \cong NO$



[Figure 45]

20. From $LP \parallel NO$, which angles are congruent and why?
21. From looking at the picture, what additional piece of information can you conclude?

22. Write a 2-column proof to prove $\triangle LMP \cong \triangle OMN$.

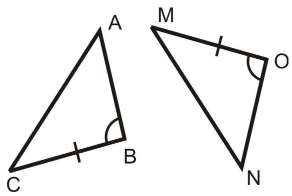
23. What would be your reason for $LM \cong MO$?

24. Fill in the blanks for the proof below. Use the given and the picture from above. Prove: M is the midpoint of PN

Statement	*Reason*
1. $LP \parallel NO, LP \cong NO$	Given
2.	Alternate Interior Angles
3.	ASA
4. $LM \cong MO$	
5. M is the midpoint of PN	

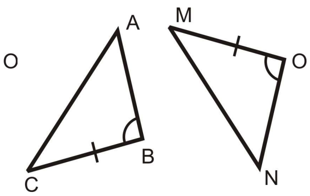
Determine the additional piece of information needed to show the two triangles are congruent by the given postulate.

AAS



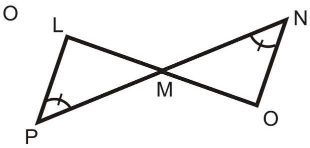
[Figure 46]

ASA



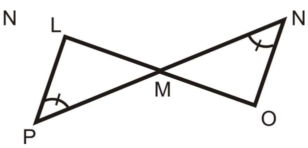
[Figure 47]

ASA



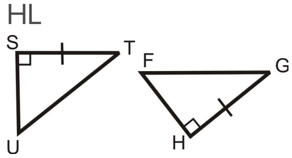
[Figure 48]

AAS



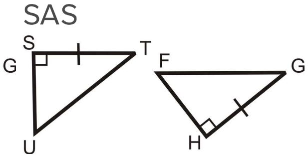
[Figure 49]

HL



[Figure 50]

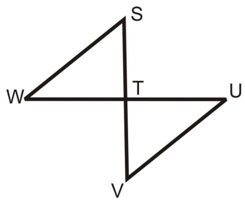
SAS



[Figure 51]

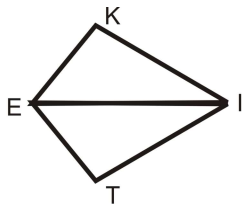
Write a 2-column proof.

Given: $SV \perp WU$ T is the midpoint of SV and WU Prove: $WS \cong UV$



[Figure 52]

Given: $\angle K \cong \angle T$, EI is the angle bisector of $\angle KET$ Prove: EI is the angle bisector of $\angle KIT$



[Figure 53]

Review Queue Answers

1.

Statement	*Reason*
1. $AD \cong DC,\ AB \cong CB$	Given
2. $DB \cong DB$	Reflexive PoC
3. $\triangle DAB \cong \triangle DCB$	SSS

2. No, only the angles are congruent, you need at least one side to prove the triangles are congruent.

3. (a) Yes, CPCTC

(b) No, these sides do not line up in the congruence statement.